day 9

From now on, with $[x]$ we mean $[x]_{\mathbb{Z}}$.
i.e. $[x] \in \mathbb{R}/\mathbb{Z}$.

we proved that $p^{-1}(\mathcal{O}_{\pm}([0]))$ is a subgroup of $\mathbb{R}$.

Goal: Theorem Every subgroup of $\mathbb{R}$ is either
1) $c\mathbb{Z}$ for some $c \geq 0$
2) dense.

$$\boxed{\text{More properties of Subgroups of } \mathbb{R}}$$

Lemma: If $G \subset \mathbb{R}$ is a subgroup, and $c \in G$ then $c\mathbb{Z} \subset G$

Proof: Step 1: we will show that for all $n \geq 1$, $cn \in G$.

Proof of step 1: By induction ..

Base: Since $c \in G$, $1 \in G$, and $c = c \cdot 1$

$$c = c \cdot 1 \in G$$

Inductive: Assume that $cn \in G$

Then $c(n+1) = cn + c \cdot 1 = cn + c$

Since $cn \in G$ & $c \in G$ by closure property of G (it is a group) we have

$$c(n+1) = cn + c \in G$$

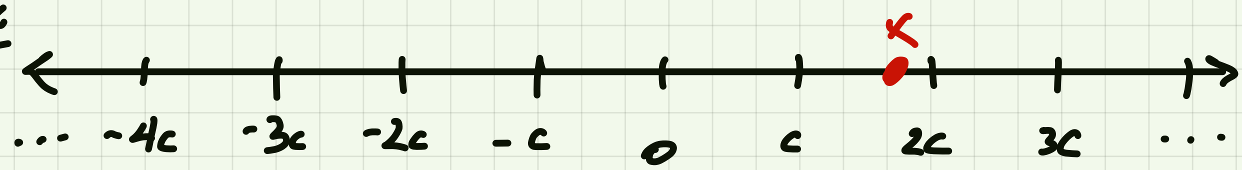
Step 2: For all $n \le -1$, $cn \in G$
Since G is a group if $cn \in G$ for $n \geq 1$
So it inverse. Therefore $-cn \in G$
and $-cn = c(-n) \in G$.

Step 3: when $n = 0$, $cn = c \cdot 0 = 0$. So
$0 = c0 \in G$, since the identity
is an element of every subgroup.

Exercise If $c \in \mathbb{R}$, $c\mathbb{Z} = (-c)\mathbb{Z}$

Lemma: If $c > 0$ and $x \in \mathbb{R}$, there exists
some $y \in c\mathbb{Z}$ such that $|x - y| < c$

ex



we can choose $y = 2c$
or $y = c$
From the picture,
$|x - y| < c$

Proof of lemma: Let n be the largest integer
such that $nc \le x$. Let $y = nc$.

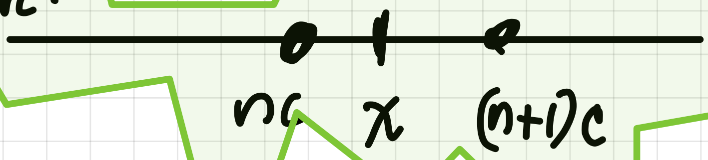
Since n was the largest integer such that
$nc \le x$, it follows that

$$x \le (n+1)c.$$

Therefore,

$$|x-y| = |x-nc| = x-nc \leq (n+1)c - nc = c$$

Picture:



Exercise Show that if $n \in \mathbb{Z}$ is the smallest integer such that $cn \geq x$, then $|x-cn| < c$

Theorem If G is not dense, then there exists $c>0$, such that $(0,c) \cap G = \phi$.

Proof we will prove this by contrapositive!

i.e. we will assume that if for every interval $(0,c)$, $G \cap (0,c) \neq \phi$, then conclude that G is dense.

Let $(a,b) \subset \mathbb{R}$ an arbitrary interval.

we will show $(a,b) \cap G \neq \phi$.

let $x = \frac{a+b}{2}$ the midpoint of (a,b)

$\varepsilon = \frac{b-a}{2}$ the distance of the mid-pt x to a and b.

Show $(a,b) \cap G \neq \phi$ is equivalent to show $(x-\varepsilon, x+\varepsilon) \cap G \neq \phi$

By assumption, there exists $\delta \in G \cap (0,\varepsilon) \neq \phi$

By today's first lemma, $\delta\mathbb{Z} < G$.

By today's second lemma, there exists $y \in \delta\mathbb{Z}$ s.t. $|x-y| < \delta$

Finally, $y \in G$ (because $\delta\mathbb{Z} < G$)

because $\delta \in G \cap (0,\varepsilon)$ then $0 < \delta < \varepsilon$.

we have that

$$|x-y| < \delta < \varepsilon.$$

That is $y \in (x-\varepsilon, x+\varepsilon)$

but $y \in G$ as well, so:

$$y \in G \cap (x-\varepsilon, x+\varepsilon) \neq \phi.$$

$\blacksquare$

Exercise Go over the proof of the theorem again.